Physics of Fluids (WBPH042-05)

8th of April 2026, 18:15-20:15

This is a closed-book exam; only simple pocket calculators are allowed. There are 3 questions in total.

When, for some reason, you are unable to answer part of questions (a, b, etc.), make a realistic assumption and use this for the rest of the question. At the end of the exam, please find an Appendix with the relevant formulae.

Write your answer to each of the 3 questions **on a separate answer sheet**. Please write your name and student number *on each answer sheet* that you hand in.

GOOD LUCK!

1 Hydraulic jump [20 pts]

Figure 1: Stilling basin

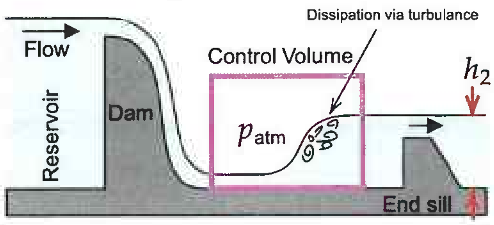


Figure 2: Hydraulic jump

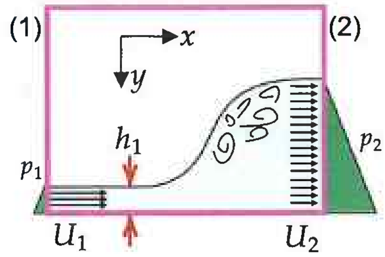


Figure 1 depicts a stilling basin; these structures are designed to reduce the velocity of water before it flows downstream, and are typically located at the outlet of a reservoir or flood storage area. When high-speed liquid discharges into a zone of lower velocity, a sudden rise in the liquid surface occurs; this phenomenon is known as a hydraulic jump. In this problem, you will solve for the steady-state height of the jump, h_2, in terms of the initial height h_1, and speed U_1 for a freshwater dam (see Figure 2). Note that for hydraulic structures, you can assume constant density and disregard the effects of viscosity, but not of gravity! See the Appendix for the relevant formulae.

(a) [5 pts] Using the static control volume depicted in Figure 2, show that

$$U_1 h_1 = U_2 h_2, \tag{1}$$

where U_2 is the final speed of the flow. Assume that the flow is uniform away from the turbulent region and the out-of-plane depth of the channel is b.

(b) [8 pts] Assuming that the pressure distribution in the channel is purely hydrostatic, show that

$$U_1^2 h_1 + \frac{g h_1^2}{2} = U_2^2 h_2 + \frac{g h_2^2}{2}. \tag{2}$$

(c) [4 pts] By solving for U_2 in Eq. (1) and substituting into Eq. (2), show that

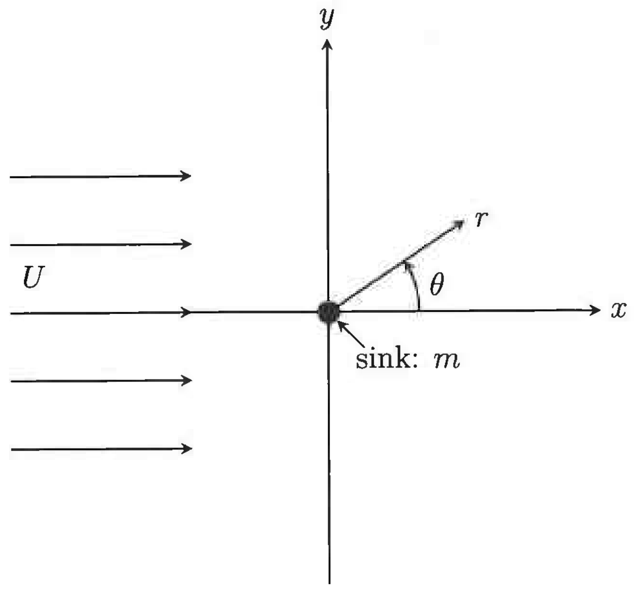
$$H^3 - (2Fr_1^2 + 1)H + 2Fr_1^2 = 0, \tag{3}$$

where $H = h_2/h_1$ (the ratio between the heights) and $Fr_1 = U_1/\sqrt{gh_1}$ is the Froude number of the channel in region 1. You do not have to solve the polynomial equation!

(d) [3 pts] If instead of holding freshwater, the dam were to hold seawater (with a different density and viscosity), but the discharge speed U_1 and height h_1 were to remain the same, what would happen to the height of the hydraulic jump h_2? Justify your answer.

2 Flow + sink [22 pts]

Consider the two-dimensional ideal flow formed by combining a uniform stream of speed U in the positive x direction and a sink of strength m at the origin.



The velocity potential for this flow is:

$$\phi = Ur\cos\theta - \left(\frac{m}{2\pi}\right)\ln r.$$

(a) [5 pts] Find the corresponding stream function. See the Appendix for the relevant formulae.

(b) [5 pts] Show that the flow described by ϕ is irrotational and incompressible.

(c) [4 pts] What are the coordinates of the stagnation point(s)?

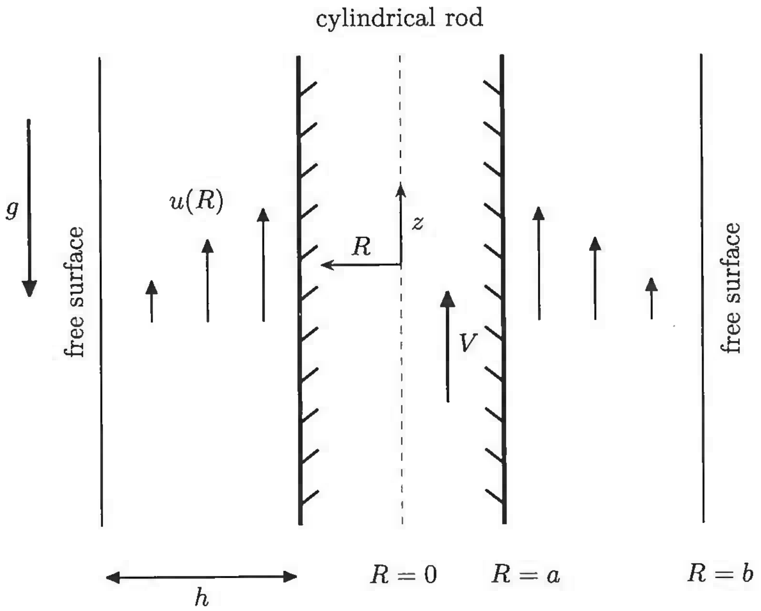
(d) [4 pts] Determine the pressure at the point $(x, y) = \left(-\frac{m}{2\pi U}, 0\right)$. The pressure and velocity far away from the origin is given by p_∞ and U, respectively.

(e) [4 pts] What happens to the stagnation point(s) if the sink is replaced with a source of the same strength at the origin?

3 Film flow on a cylindrical rod [18 pts]

A vertical cylindrical rod of radius a is pulled upward at constant speed V from a large container of viscous liquid. In steady state, a liquid film of thickness h coats the outside of the rod, so the liquid occupies the annular region

$$a \leq R \leq b.$$



Use cylindrical coordinates (R, z), where z is the axial coordinate pointing upward and R is the radial coordinate measured outward from the rod axis. See the Appendix for the relevant formulae.

Assume that the flow is steady, laminar, incompressible, axisymmetric, fully developed, and Newtonian. The pressure is constant everywhere and equal to the atmospheric pressure. Air exerts negligible shear stress on the free surface $R = b$.

The only non-zero velocity component is the axial velocity $u(R)$, so that:

$$\boldsymbol{u} = u(R)\,\boldsymbol{e}_z.$$

(a) [8 pts] Show that the axial velocity distribution is:

$$u(R) = V + \frac{g}{4\nu}(R^2 - a^2) - \frac{gb^2}{2\nu}\ln\left(\frac{R}{a}\right), \qquad \nu = \frac{\mu}{\rho}.$$

(b) [4 pts] Show that the minimum of $u(R)$ is attained at the free surface $R = b$.

(c) [6 pts] Next, we analyze two different fluids:

- liquid mercury: $\rho = 13\,590$ kg m^{-3}, $\mu = 0.0015$ Pa s,
- castor oil: $\rho = 952$ kg m^{-3}, $\mu = 0.650$ Pa s.

Compare the shear stress at the rod surface, and the velocity at the free surface, for fixed a, b, and V. Briefly interpret the result.

Appendix

Conservation of mass for a control volume:

$$\frac{d}{dt}\left[\int_{V^*(t)} \rho dV\right] + \oint_{A^*(t)} \rho\,[\boldsymbol{u}-\boldsymbol{b}]\cdot\boldsymbol{n} dA = 0$$

Conservation of momentum for a control volume:

$$\frac{d}{dt}\left[\int_{V^*(t)} \rho \boldsymbol{u} dV\right] + \oint_{A^*(t)} [\rho\boldsymbol{u}]\,[\boldsymbol{u}-\boldsymbol{b}]\cdot\boldsymbol{n} dA = \int_{V^*(t)} \rho \boldsymbol{g} dV + \oint_{A^*(t)} \boldsymbol{f}(\boldsymbol{n}) dA$$

For an incompressible, axisymmetric flow, the velocity field is:

$$\boldsymbol{u} = u_R(R,z)\,\boldsymbol{e}_R + u_z(R,z)\,\boldsymbol{e}_z.$$

The continuity equation is:

$$\frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{\partial u_z}{\partial z} = 0.$$

The radial momentum equation is:

$$\rho\left(u_R\frac{\partial u_R}{\partial R} + u_z\frac{\partial u_R}{\partial z}\right) = -\frac{\partial p}{\partial R} + \mu\left[\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R}\frac{\partial u_R}{\partial R} - \frac{u_R}{R^2} + \frac{\partial^2 u_R}{\partial z^2}\right].$$

The axial momentum equation is:

$$\rho\left(u_R\frac{\partial u_z}{\partial R} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right) + \frac{\partial^2 u_z}{\partial z^2}\right] - \rho g.$$

If the radial velocity vanishes and the axial velocity depends only on R, the only non-zero shear stress is:

$$\tau_{Rz} = \mu\frac{du}{dR}.$$

For a velocity potential $\phi(r,\theta)$, the velocity components are given by:

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

The z-component of the vorticity in 2D is:

$$\omega_z = (\nabla \times \mathbf{V})_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

For a two-dimensional velocity field $\mathbf{V} = (u_r, u_\theta)$, the divergence is:

$$\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

For a scalar field $\phi(r,\theta)$, the Laplacian in polar coordinates is:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$